Mathematical Induction

June 10, 2018

1 Introduction

You've seen one important proof technique: mathematical induction. We want to prove something for a general statement $P(n)$. According to *The principle of induction*, you can prove $P(n)$ is true for all $n \in \mathbb{N}$ using the following three steps:

(i) **Base Case**: Prove that $P(0)$ is true.

(ii) **Inductive Hypothesis**: For arbitrary $k \geq 0$, assume that $P(k)$ is true.

(iii) **Inductive Step**: With the assumption of the Inductive Hypothesis in hand, show that $P(k+1)$ is true.

Side note: the second step is redundant, in the sense that it is captured by the third step. Personally, I use two steps: base case (proving P(0)) and inductive step (P(k) $\implies$ P(k+1)).

Hopefully, the dominoes analogy on the course notes makes sense: imagine you have dominoes numbered $0, 1, \cdots$. Proving $P(k) \implies P(k+1)$ corresponds to "if the k^{th} domino is knocked over, then the $k+1^{st}$ domino would be knocked over", and $P(0)$ corresponds to "I will knock over the first domino". Thus, this chain reaction will knock down all dominoes, showing that the statement is true for all n.

But mathematically, *why*?

2 Mathematical Induction

Let's begin by taking a closer look at the set of natural numbers:

$$\mathbb{N} = \{0, 1, 2, \cdots\}$$

, the set of positive integers plus 0. Each positive integer n has a successor $n+1$. $\mathbb{N}$ has lots of nice properties, and some of them are listed below:

(i) $0 \in \mathbb{N}$.

(ii) If $n \in \mathbb{N}$, then its succesor $n+1 \in \mathbb{N}$.

(iii) 0 is not the successor of any element in $\mathbb{N}$.

(vi) If $n, m \in \mathbb{N}$ have the same successor, then $n = m$.

(v) A subset of $\mathbb{N}$ which contains 0, and which contains $n+1$ whenever it contains n, must equal $\mathbb{N}$.

Let's focus on the last property, which closely resembles mathematical induction, and prove that it holds.

Proof. Suppose the statement is false. Then $\mathbb{N}$ contains a set S such that

(i) $0 \in S$

(ii) If $n \in S$, then $n+1 \in S$

and yet $S \neq \mathbb{N}$. Consider the smallest member of the set $\{n \in \mathbb{N} : n \notin S\}$, call it n_0. Since (i) holds, it is clear $n_0 \neq 0$. So n_0 is a successor to some number in $\mathbb{N}$, namely $n_0 - 1$. We have $n_0 - 1 \in S$ since n_0 is the smallest number of $\{n \in \mathbb{N} : n \notin S\}$. By (ii), the successor of $n_0 - 1$, namely n_0 is also in S, which is a contradiction because we know in the first place that $n_0 \notin S$. $\square$

The last property is the basis of mathematical induction. Let $P_0, P_1, P_2, \cdots$ be a list of statements or propositions that may or may not be true. The principle of mathemetical induction asserts all the statements $P_0, P_1, P_2, \cdots$ are true provided

(i) **Base Case**: P_0 is true,

(ii) **Inductive Step**: P_{n+1} is true whenever P_n is true.

3 Reference

The first section refered to CS70 Note 0 (http://www.eecs70.org/static/notes/n0.pdf). The second section is copied from Chapter 1 of *Elementary Analysis: The Theory of Calculus, 2nd edition* by Kenneth A. Ross, modified with adding 0 to $\mathbb{N}$.