Topic : Trees & planar graphs

Trees   minimal connected graphs

- Definitions (equivalent)

① connected & acyclic   intuitive definition

② connected & $|V|-1$ edges   what's the number of edges ?

③ connected & removal of any edge disconnects it   what happens if I remove an edge ?

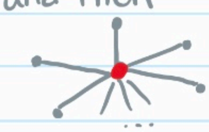
④ any new edge creates a cycle & acyclic (connected &)   what happens if I add an edge ?

- A leaf is a vertex of degree one.   It is a special vertex.

Q1 This question explores max and min # leaves of a tree with more than 3 vertices

(a) What is the largest possible number of leaves the tree could have ?

Sol: [The idea is, you can always pick a center vertex, and then attach the rest of the vertices as leaves. So, the maximum number of leaves is $n-1$. Now, let's prove it.]



Maximum number of leaves is $n-1$.

Step 1 (show $n-1$ leaves is possible to achieve):

Pick a vertex. Then attach the rest of the vertices to it. We've created a graph with $n-1$ leaves.

Step 2 (there cannot exist a tree with more than $n-1$ leaves)

[This step show that $n-1$ is the maximum.]

Suppose there exists a tree that has more than $n-1$ leaves.

$\Rightarrow$ all vertices have degree 1

$\Rightarrow$ all vertices only have one neighbor

Pick a vertex x with unique neighbor y.

$\Rightarrow$ x, y form a connected component separate with the rest of the tree.

Contradiction, as all trees are connected.

Thus, there cannot exist a tree that has more than $n-1$ leaves. $\square$

Planarity: [It's hard to check whether a graph is planar or not just by looking at it. However, it's much easier to count the number of edges / vertices of a graph.]

<u>connected</u> planar graph $\Rightarrow v + f = e + 2$

G is planar $\Rightarrow$ G can be colored using 4 colors

G is planar $\Rightarrow e \leq 3v - 6$

A bipartite graph is planar $\Rightarrow e \leq 2v - 4$

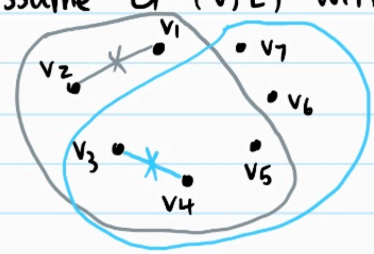
use contrapositive to prove a graph is nonplanar e.g. K_5, $K_{3,3}$

Q3

Property T: $\forall v_1, v_2, v_3 \in V$, $v_1 \neq v_2 \neq v_3$, there are at least two edges among them.

Prove $|V| \geq 7$ and G has property T $\Rightarrow$ G is nonplanar

<u>Pf:</u> Assume $G = (V, E)$ with $|V| = 7$ is planar.

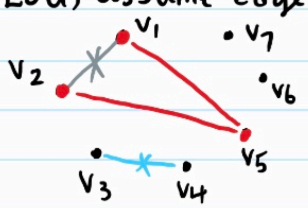


$\{v_1, v_2, v_3, v_4, v_5\}$ cannot form K_5.

WLOG, assume edge $\{v_1, v_2\} \notin E$.

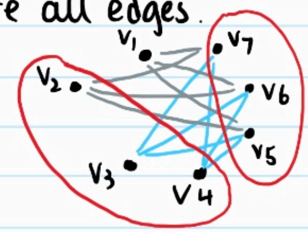
$\{v_3, v_4, v_5, v_6, v_7\}$ cannot form K_5.

WLOG, assume edge $\{v_3, v_4\} \notin E$



Take a look at v_1, v_2, v_5. According to property T, $\{v_1, v_5\}, \{v_2, v_5\} \in E$.

Similarly, $\{v_2, v\}$, $\{v_3, v\}$, $\{v_4, v\}$ where $v \in \{v_5, v_6, v_7\}$ are all edges.



Thus, any three vertices from $\{v_1, v_2, v_3, v_4\}$, with $\{v_5, v_6, v_7\}$ form an instance of $K_{3,3}$. Contradiction.

Therefore, all graphs with 7 vertices and property T is non-planar.

$\Rightarrow$ all graphs with ≥ 7 vertices and property T is non-planar.

Bipartite graphs

Q2

G is bipartite $\Leftrightarrow$ G has no tour of odd length

<u>Pf</u>: To show two propositions are equivalent ($P \Leftrightarrow Q$) you need to show ① $P \Rightarrow Q$ and ② $Q \Rightarrow P$.

For bipartite graphs, it's easier to think of the two sets of vertices are 2 colors.

($\Rightarrow$) Assume G is bipartite. WTS G has no tour of odd length.

G is bipartite $\Rightarrow$ we can separate V into $\underset{\text{left}}{L}$ and $\underset{\text{right}}{R}$ such that there's no edge within L or R.

WLOG, start the tour at a vertex in L.

1^{st} edge takes us from L to R

2^{nd} edge takes us from R to L

$\vdots$

$2n^{th}$ edge takes us from R to L

$(2n+1)^{st}$ edge takes us from L to R

Since the tour must end in L, the tour length must be even.

($\Leftarrow$) Assume G has no tours of odd length. WTS G is bipartite.

Let $v \in V$.

$R = \{$ vertices s.t. shortest path to v is odd $\}$

$L = \{$ vertices s.t. shortest path to v is even $\}$

- If edge $\{u_1, u_2\} \in E$ where $u_1 \in R, u_2 \in R$, then we've found a tour with odd length as follows:

  $v \xrightarrow[\text{odd}]{} u_1 \xrightarrow[1]{} u_2 \xrightarrow[\text{odd}]{} v$

  $\Rightarrow$ no vertices in R are connected

- Similarly, if $\{v_1, v_2\} \in E$ where $u_1 \in L, u_2 \in L$, then we've found a tour w/ odd length:

  $v \xrightarrow[\text{even}]{} v_1 \xrightarrow[1]{} v_2 \xrightarrow[\text{even}]{} v$

  $\Rightarrow$ no vertices in L are connected

Repeat the above process for each connected components.

$\Rightarrow$ G is bipartite

$\blacksquare$

with more than 2 vertices

(b) Prove that every tree must have at least two leaves.

<u>Sol</u>: [You can find a path between two leaves. However, you cannot use this fact yet, since you haven't proven the existence of two leaves in a tree.

Surely, since a tree is connected, you can find a path between <u>any</u> pair of vertices.

In order for a pair of vertices to be leaves, you hope to find two vertices such that you cannot attach other vertices to them. What prevents you from "connecting an extra vertex" to the end points? This motivates the idea of looking at the longest path.]

Consider the longest path $\{v_0, v_1\}, \ldots, \{v_{k-1}, v_k\}$.

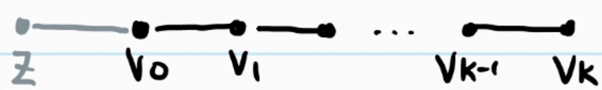
We want to show that v_0 and v_k must be leaves.

WLOG, ← without loss of generality. In this case, this means it doesn't matter if we look at v_0 or v_k ... they're essentially the sa

suppose v_0 is not a leaf.

$\Rightarrow \deg(v_0) > 1$

$\Rightarrow$ v_0 connects to another vertex z that is different from v_1.



$\Rightarrow$ we can add edge $\{z, v_0\}$ to the path to get a longest path.

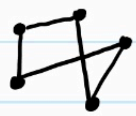
Contradiction, since we assumed in the beginning that $\{v_0, v_1\} \ldots \{v_{k-1}, v_k\}$ is the longest path.

$\square$

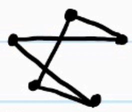
Q4

Define the complement of G as $\bar{G} = (V, \bar{E})$ where $\bar{E} = (V \times V) - E$.

example:



$G = (V, E)$



$\bar{G} = (V, \bar{E})$

(a) Suppose G has v vertices, e edges. How many vertices and edges does $\bar{G}$ have?

Sol: maximum number of edges $= \frac{n(n-1)}{2}$ (pick a vertex; pick another vertex) $\leftarrow \{v_i, v_j\} = \{v_j, v_i\}$

$|\bar{E}| = \frac{n(n-1)}{2} - |E| = \frac{n(n-1)}{2} - e$

$|V| = v$

(b) Prove that $v \geq 13$ and G planar $\Rightarrow \bar{G}$ nonplanar

Sol: A comman strategy to show a graph is nonplanar is by showing $|\bar{E}| > 3v - 6$.

G is planar $\Rightarrow |E| \leq 3v - 6$

$\overset{(a)}{\Longrightarrow} |\bar{E}| > \frac{n(n-1)}{2} - (3v-6) = \underbrace{\frac{n(n-1)}{2} - 3v + 6}$

WTS ↑ is greater than $3v-6$

$v \geq 13 \Rightarrow \frac{v(v-1)}{2} \geq \frac{v \cdot 12}{2} = 6v$

$\Rightarrow |\bar{E}| > 6v - 3v + 6 = 3v + 6 > 3v - 6$

Thus, $\bar{G}$ must be planar.

$\square$

(c) True / False : $\bar{G}$ non-planar $\Rightarrow G$ planar ?

Sol: [Converse is true only if all implications are reversible in (b). Notice that $|E| \leq 3v-6 \not\Rightarrow G$ is planar, We can start with a nonplanar graph, and then "force" $|E| \leq 3v-6$ by adding dummy vertices. This leads to the counterexample below.]

False.

← any five of the isolated vertices form a K_5 in $\bar{G}$.

G