## Coloring

- planar $\Rightarrow$ 4 colorable    4 colorable $\not\Leftarrow$ planar

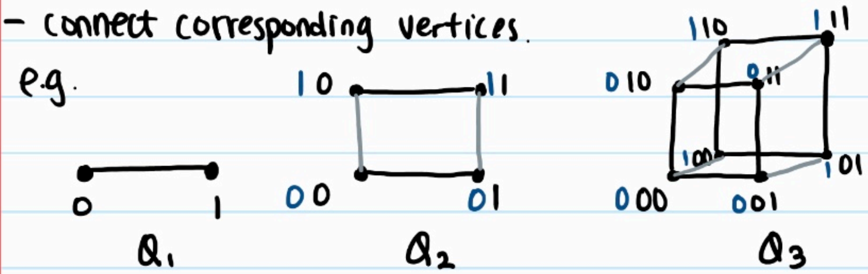
bipartite $\Leftrightarrow$ 2 colorable

## Hypercubes

- How to get $Q_d$?

- Get two copies of $Q_{d-1}$.
- Add 0 to one copy of the vertices.
- Add 1 to the other copy.
- connect corresponding vertices.

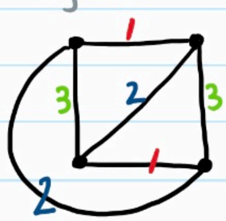
e.g.




Q1 Let's color edges instead :)

(a) Show $K_4$ can be 3 edge colored.

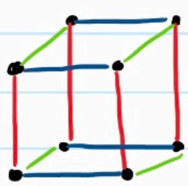
<u>Sol</u>: Try it!




(b) How many colors are required to edge color $Q_3$?

<u>Sol</u>: [Put $Q_3$ in coordinates. Notice that no matter which edge I pick, the other two edges (pointing at the other two dimensions) have to be colored using 2 different colors.
There's no constrains for edges that are in the same dimension]
vertex

3.



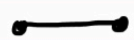
edge

(c) Prove that any graph with maximum degree d is 2d-1 colorable.

<u>Sol</u>: Trying to coloring the entire graph would be a headache, thus induction seems like a good approach.
If we use induction, we can isolate a vertex/edge, get a (partially) colored graph using IH (this means, our hypothesis does MOST of the work for us).
What do we want to do induction on ?
① d (max degree)? It's hard to reduce a graph from max degree d+1 to max degree d.
② v (# vertices)? We're coloring edges here. Removing a vertex might cause removal of several edges, which would cause complexity.
③ e (# edges)? Seems reasonable. Let's try it.

<u>P(n)</u>: If G has n edges, then G is 2d-1 colorable.

<u>Base case</u>: P(1) is true.



This graph is 2x1-1 colorable.

<u>Inductive Step</u>: WTS P(n) ⇒ P(n+1).

Let G be a graph w/ n+1 edges. $\boxed{\text{Always}}$ start with graph in P(n+1) to avoid

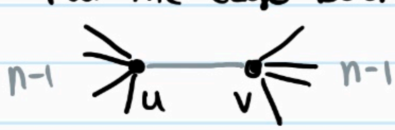
Remove an edge in G to get G'.

⇒ G' has n edges

⇒ G' is 2d'-1 colorable where d' is max degree in G'

d' ≤ d ⇒ G' is 2d-1 colorable

Put the edge back.




deg(u) ≤ d ⇒ u is incident to at most d-1 other edges.

Similarly, v " " " .

⇒ at most 2n-2 colors are unavailable to edge {u, v}.

⇒ G is 2n-1 colorable.

□

edge

(d) $G$ is a tree $\Rightarrow$ $d$ colorable

max degree

**Sol:** [ What's special about trees? Notice that if I remove an edge from a tree, I wouldn't get a tree back, and then we can't really use "$d$ colorable" in $G'$ since the implication would be vacuously tree in that case. However, if I remove a leaf (vertex) from $G$, I would still get a tree back. ]

<u>$P(n)$</u>: $G$ is a tree w/ $n$ vertices $\Rightarrow$ $d$ colorable

<u>Base case</u>: $P(1)$ is true.

0 colorable, since no edges to be colored.

<u>IS</u>: Assume $P(n)$. WTS $P(n+1)$ holds.

Let $G$ be a graph with degree $n+1$.

Remove a leaf $v$ and call the remaining graph $G'$.

$\Rightarrow G'$ is $d$ colorable


$\deg(u) \leq d \Rightarrow u$ is incident to at most $d-1$ other vertices

Add $v$ back. There's at least one available color for edge $\{u,v\}$

$\Rightarrow G$ is $d$ colorable.

$\square$