Understanding Representation of Symmetric Groups via Young Tableaux

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1 Introduction

In representation theory, we study groups by associating linear objects to group elements. Given a group G, we can pick a vector space and choose some invertible endomorphisms of the vector space to "represent" the elements of G, so that the group operation is preserved under this identification. More formally, a *representation* of a group G is linear group actions of G on a vector space V over a field F. Different choices of G, V and F give us different areas to investigate. A simpler case to start with is when G is a finite group, V is a finite-dimensional, and F is $\mathbb{C}$. As we shall see, in this case, any representation can be decompose into a direct sum of irreducible ones. Two important questions in representation theory are:

1. *how to construct the irreducible representations of G?*
2. *how to completely reduce a given representation G?*

Although these two questions are hard to answer for an arbitrary group G, we will illustrate the answers for G being a symmetric group with the help of *Young tableaux*.

In section 2, we introduce basics of representation theory of finite groups. In section 3, we introduce Young tableaux, a combinatorial object that is used in representation theory to study the representation of S_n and $\mathrm{GL}(n)$. In secion 4, we give the correspondence between irreducible representations of S_n and Young diagrams. Section 5 serves as guides to future studies, which introduces studying Grassmannians via Young diagrams. This article assumes basic understanding of linear algebra and abstract algebra at an undergraduate level.

2 Representations of Finite Groups

2.1 Representation and G-module

Representations give us a way to study groups via linear objects. Below are two (equivalent) ways of associating linear objects with group elements. In this article, all vector spaces are over $\mathbb{C}$.

Definition 2.1.1 (representation)**.** Let G be a group. A representation of G is a group homomorphism $\varphi : G \to \mathrm{GL}(V)$ for some vector space V. When the homomorphism is assumed, we say V is a representation of G and denote $\varphi(g)$ by $g|_V$.

Definition 2.1.2 (G-module)**.** Let G be a group acting on a vector space V. We say V is a G-module if $g \cdot (c_1 v_1 + c_2 v_2) = c_1(g \cdot v_1) + c_2(g \cdot v_2)$ for all $g \in G$, $v_1, v_2 \in V$ and $c_1, c_2 \in \mathbb{C}$, i.e., the group action is linear.

Given a G-module V, the map $\varphi : G \to \mathrm{End}(V)$ given by $\varphi(g)(v) = g \cdot v$ is a homomorphism; moreover, since

$$\mathrm{id}_V = \varphi(1) = \varphi(gg^{-1}) = \varphi(g)\varphi(g^{-1}),$$

we have $\varphi(g)^{-1} = \varphi(g^{-1})$, i.e., $\varphi(g) \in \mathrm{GL}(V)$. On the other hand, given a representation $\varphi : V \to \mathrm{GL}(V)$, G acts on V via $g \cdot v = \varphi(g)(v)$; this group action is linear since

$$g \cdot (c_1 v_1 + c_2 v_2) = \varphi(g)(c_1 v_2 + c_2 v_2) = c_1 \varphi(g)(v_1) + c_2 \varphi(g)(v_2) = c_1(g \cdot v_1) + c_2(g \cdot v_2).$$

Hence, V being a representation of G is the same as V being a G-module. We will use these two views interchangeably. In this article, we will only discuss the case where G is a finite group and V is a finite dimensional complex vector space.

Example 2.1.3. Consider $G = S_2$. Notice that $\varphi : S_2 \to \mathrm{GL}(\mathbb{C}^2)$ given by

$$1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is a group homomorphism.

Equivalently, S_2 acts on $\mathbb{C}^2$ linearly by

$$\sigma((x_1, x_2)) = (x_{\sigma(1)}, x_{\sigma(2)}) \text{ for } \sigma \in S_2$$

i.e. $\mathbb{C}^2$ is a S_2-module. This is called the *permutation representation* of S_2.

Example 2.1.4. Given finite group G, let $\mathbb{C}[G]$ denote the vector space freely generated by elements of G. The action of G on itself by left translation extends to a linear action of G on $\mathbb{C}[G]$. This is called the *regular representation* of G.

Example 2.1.5. Consider $G = S_3$ with subgroup $H = \{e, (1\ 2)\}$. Let $\mathcal{H} = \{eH, (1\ 3)H, (2\ 3)H\}$. Then, H acts on $\mathbb{C}[\mathcal{H}]$ by $h \cdot gH = (hg)H$. This is called the *coset representation of G with respect to H*.

2.2 Complete Reducibility

We want to formalize the simplest G-modules:

Definition 2.2.1 (irreducibility)**.** Let V be a nonzero G-module. We say V is irreducible if there is no proper subspace $W \neq 0$ that is invariant under the action of G.

Example 2.2.2. The S_2-module $\mathbb{C}^2$ in Example 2.1.3 is not irreducible, as $\{(x, x) : x \in \mathbb{C}\}$ is a nonzero proper S_2-invariant subspace of $\mathbb{C}^2$.

Not all reducible G-modules can be decomposed into a direct sum of irreducible ones. However, it is the case if G is a finite group. Therefore, in order to understand an arbitrary representation of G, complete reducibility of representations of finite groups gives us motivations to study irreducible representations.

Theorem 2.2.3 (Maschke's Theorem)**.** *Let G be a finite group and let V be a nonzero G-module. Then $V = V_1 \oplus \cdots \oplus V_n$ where $V_1, \ldots, V_n$ are irreducible G-submodule.*

Proof. Suppose V is not irreducible and W is a nonzero proper G-submodule of V. We will prove by inducting on $n = \dim V$.

Let $\{v_1, \ldots, v_n\}$ be a basis of V. Given $v = c_1v_1 + \cdots + c_nv_n$ and $w = d_1v_1 + \cdots + d_nv_n$, consider the inner product given by

$$\langle v, w\rangle = c_1\overline{d_1} + \cdots + c_n\overline{d_n}.$$

Now, define another inner product as

$$\langle v, w\rangle' = \sum_{g\in G} \langle g \cdot v, g \cdot w\rangle.$$

Notice that $\langle v, w\rangle'$ is G-invariant: for $h \in G$, we have

$$\langle h \cdot v, h \cdot w\rangle' = \sum_{g\in G} \langle g \cdot (h \cdot v), g \cdot (h \cdot v)\rangle = \sum_{g\in G} \langle (gh) \cdot v, (gh) \cdot w\rangle = \sum_{g\in G} \langle g \cdot v, g \cdot w\rangle = \langle v, w\rangle'.$$

Let $W^\perp = \{v \in V : \forall w \in W, \langle v, w\rangle' = 0\}$. Since $V = W \oplus W^\perp$ as vector spaces and W is a G-submodule, in order to show V decomposes into W and $W^\perp$ as G-modules, we need to show $W^\perp$ is a G-module: let $g \in G, u \in W^\perp$. For all $w \in W$,

$$\langle g \cdot u, w\rangle' = \langle g^{-1} \cdot (g \cdot u), g^{-1} \cdot w\rangle' = \langle u, g^{-1} \cdot w\rangle' = 0 \text{ i.e. } g \cdot u \in W^\perp.$$

Since $\dim W, \dim W^\perp < n$, by induction, W and $W^\perp$ can be written as a direct sum of irreducibles. Hence, $V = W \oplus W^\perp$ decomposes into a direct sum of irreducibles. $\square$

2.3 G-homomorphism

We want a notion of isomorphisms between G-modules; or more generally, homomorphisms between G-modules.

Definition 2.3.1 (G-homomorphism; G-isomorphism; G-isomorphic)**.** Let V, W be G-modules. We say a linear transformation $\varphi : V \to W$ is a G-homomorphism if $\varphi(g \cdot v) = g \cdot \varphi(v)$ for all $g \in G, v \in V$. If in addition φ is bijective, we say φ is a G-isomorphism and write $V \cong W$.

Note that if $\varphi : V \to W$ is a G-homomorphism, then $\ker\varphi$ is a G-submodule of V and $\mathrm{Im}\varphi$ is a G-submodule of W.

Lemma 2.3.2 (Schur's Lemma)**.** *Let $\varphi : V \to W$ be a G-homomorphism between irreducible G-modules. Then either φ is a G-isomorphism or φ is the zero map. In particular, if $V \cong W$, then $\varphi = \lambda I$ for some $\lambda \in \mathbb{C}$.*

Proof. Since V and W are irreducible, either $\ker\varphi = \{0\}$ and $\mathrm{Im}\varphi = W$, or $\ker\varphi = V$ and $\mathrm{Im}\varphi = \{0\}$. The former means φ is bijective and the later means φ is zero.

Suppose $V \cong W$. Since $\mathbb{C}$ is algebraically closed, there exists eigenvalue $\lambda \in \mathbb{C}$ of linear transformation φ. Since $\varphi - \lambda I$ is a non-injective G-homomorphism, we must have $\varphi - \lambda I = 0$. □

Lemma 2.3.3. *Let V, W be G-modules. Then $V^* \otimes W \cong \mathrm{Hom}(V, W)$ as G-modules, where G acts on V^* linearly by $(g \cdot \phi)(v) = \phi(g^{-1} \cdot v)$ and G acts on $V^* \otimes W$ linearly by $g \cdot (\phi \otimes w) = (g \cdot \phi) \otimes (g \cdot w)$.*

Proof. Let $\varphi \in \mathrm{Hom}(V, W)$. Then G acts on $\mathrm{Hom}(V, W)$ linearly by $(g \cdot \varphi)(v) = g \cdot (\varphi(g^{-1} \cdot v))$, i.e., $\mathrm{Hom}(V, W)$ is a G-module.

Consider a vector space isomorphism $f : V^* \otimes W \to \mathrm{Hom}(V, W)$ given by

$$f(\phi \otimes w)(v) = \phi(v)w.$$

Notice that f is also a G-homomorphism, since

$$f(g \cdot (\phi \otimes w))(v) = f((g \cdot \phi) \otimes (g \cdot w))(v) = (g \cdot \phi)(v)(g \cdot w) = \phi(g^{-1} \cdot v)(g \cdot w)$$

$$(g \cdot f(\phi \otimes w))(v) = g \cdot (f(\phi \otimes w)(g^{-1} \cdot v)) = g \cdot (\phi(g^{-1} \cdot v)w) = \phi(g^{-1} \cdot v)(g \cdot w).$$

□

2.4 Character

In this section, we will see representations of a finite group are determined by their characters, in the sense that characters of irreducible representations form a orthonormal basis of a vector space (which we shall define).

Definition 2.4.1 (character)**.** Let V be a G-module. Then the character $\chi_V : G \to \mathbb{C}$ of V is defined by $\chi_V(g) = \mathrm{Tr}(g|_V)$.

Example 2.4.2. Consider the representation of S_2 on $\mathbb{C}^2$ in Example 2.1.3. We have

$$\chi_V(1) = \mathrm{Tr}(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 2,$$

$$\chi_V(\begin{pmatrix} 1 & 2 \end{pmatrix}) = \mathrm{Tr}(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) = 0.$$

Lemma 2.4.3. *Let V, W be G-modules. Then $\chi_{V\oplus W} = \chi_V + \chi_W$, $\chi_{V\otimes W} = \chi_V \cdot \chi_W$ and $\chi_{V^*} = \overline{\chi_V}$.*

Proof. Let $g \in G$. Then,

$$\chi_{V\oplus W}(g) = \mathrm{Tr}(g|_{V\oplus W}) = \mathrm{Tr}(\begin{bmatrix} g|_V & 0 \\ 0 & g|_W \end{bmatrix}) = \chi_V(g) + \chi_W(g).$$

Suppose $\{v_1, \ldots, v_m\}, \{w_1, \ldots, w_n\}$ are bases of V, W with corresponding dual bases $\{v_1^*, \ldots, v_m^*\}$, $\{w_1^*, \ldots, w_n^*\}$. Notice that for any G-module U with basis $\{u_i\}$ and dual basis $\{u_i^*\}$, we have $\chi_U(g) = \sum_i u_i^*(g \cdot u_i)$. Then

$$\begin{aligned}\chi_{V\otimes W}(g) &= \sum_{i=1}^{m}\sum_{j=1}^{n}(v_i^* \otimes w_j^*)(g \cdot v_i \otimes g \cdot w_j) = \sum_{i=1}^{m}\sum_{j=1}^{n} v_i^*(g \cdot v_i) w_j^*(g \cdot w_j) \\ &= \sum_{i=1}^{m} v_i^*(g \cdot v_i) \sum_{j=1}^{n} w_j^*(g \cdot w_j) = \mathrm{Tr}(g|_V) \cdot \mathrm{Tr}(g|_W) = \chi_V(g) \cdot \chi_W(g).\end{aligned}$$

Suppose $\{\lambda_1, \ldots, \lambda_n\}$ are eigenvalues of $g|_V$. Since $(g|_V)^{|G|} = 1$, $\lambda_i's$ are units of unity, i.e., $\lambda_i^{-1} = \overline{\lambda_i}$. Recall G acts on V^* linearly by $(g \cdot \phi)(v) = \phi(g^{-1} \cdot v)$, then

$$\chi_{V^*}(g) = \chi_V(g^{-1}) = \mathrm{Tr}((g|_V)^{-1}) = \sum_{i=1}^{n} \lambda_i^{-1} = \sum_{i=1}^{n} \overline{\lambda_i} = \overline{\mathrm{Tr}(g|_V)} = \overline{\chi_V(g)}.$$

□

Definition 2.4.4 (class function)**.** A map $f : G \to \mathbb{C}$ is a *class function* if $f(g_1) = f(g_2)$ whenever g_1, g_2 are in the same conjugacy class. We denote the $\mathbb{C}$-vector space of class functions by $\mathbb{C}_{class}(G)$.

Theorem 2.4.5. *Let V be a representation of G. Then, $\chi_V \in \mathbb{C}_{class}(G)$.*

Proof. Suppose g_1, g_2 are in the same conjugacy class. There exists $h \in G$ such that $g_1 = hg_2h^{-1}$. Then we have

$$\chi_V(g_1) = \chi_V(hg_2h^{-1}) = \mathrm{Tr}((hg_2h^{-1})|_V) = \mathrm{Tr}((h|_V)(g_2|_V)(h|_V)^{-1}) = \mathrm{Tr}(g_2|_V) = \chi_V(g_2).$$

□

	1	1
S_2	1	$\begin{pmatrix}1 & 2\end{pmatrix}$
U_2	1	1
U_2'	1	-1

Table 1: Character Table of S_2

Example 2.4.6. Let S_2 act on $\mathbb{C}$. Then, any linear action induces an irreducible representation since $\mathbb{C}$ does not have nontrivial subspace. The *trivial representation* U_2 of S_2 is given by $g \mapsto 1$; the *alternating representation* V_2 of S_2 is given by $g \mapsto \text{sign}(g)$. Hence, we construct the character table by using each column to denote each conjugacy class, and putting the size of conjugacy class on top:

In Example 2.1.3, the *permutation representation* $\mathbb{C}^2$ decomposes into a direct sum of irreducible representations:

$$\mathbb{C}^2 = \{(x, x) : x \in \mathbb{C}\} \oplus \{(x_1, x_2) : x_1 + x_2 = 0\}.$$

Notice that $\{(x, x) : x \in \mathbb{C}\} \cong U_2$ and $\{(x_1, x_2) : x_1 + x_2 = 0\} \cong U_2'$. The above table contains all irreducible representations of S_2.

Definition 2.4.7. Define a inner product on $\mathbb{C}_{class}(G)$ by $\langle \phi_1, \phi_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\phi_1(g)} \phi_2(g)$.

In the example 2.4.6, notice that $\langle \chi_{U_2}, \chi_{U_2} \rangle = \langle \chi_{U_2'}, \chi_{U_2'} \rangle = 1$ and $\langle \chi_{U_2}, \chi_{U_2'} \rangle = 0$, i.e., χ_{U_2} and $\chi_{U_2'}$ are orthonormal. This observation holds in general:

Theorem 2.4.8. *Under the inner product defined above, the characters of irreducible representations of G are orthonormal in the vector space $\mathbb{C}_{class}(G)$.*

We will first prove the following lemma about the dimension of G-invariants.

Lemma 2.4.9. *Let $V^G = \{v \in V : \forall g \in G, gv = v\}$ be the set of G-invariants of V. Then $\dim(V^G) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$.*

Proof. Consider $\pi : V \to V$ given by

$$v \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot v.$$

We can check that π is a G-module homomorphism. Moreover, π is a projection from V to V^G since $\pi(u) = u$ for all $u \in V^G$ and $\pi(\pi(v)) = \pi(v)$ for all $v \in V$.

Since the trivial representation is one-dimensional, the multiplicity of the trivial representation is the same as $\dim(V^G)$. Notice that π is the identity map on V^G and kills other elements in V, hence:

$$\dim(V^G) = \text{Tr}(\pi|_V) = \text{Tr}(\frac{1}{|G|} \sum_{g \in G} g|_V) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

□

Proof of Theorem 2.4.8. Let V, W be irreducible G-modules. By Lemma 2.3.3, $\mathrm{Hom}(V, W)$ is also a G-module. Consider $\mathrm{Hom}(V, W)^G$, the set of G-module homomorphisms from V to W. By Lemma 2.4.9,

$$\dim(\mathrm{Hom}(V, W)^G) = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Hom}(V,W)}(g).$$

By Lemma 2.3.3 and Lemma 2.4.3, the equation becomes

$$\dim(\mathrm{Hom}(V, W)^G) = \frac{1}{|G|} \sum_{g \in G} \chi_{V^* \otimes W}(g) = \frac{1}{|G|} \sum_{g \in G} \chi_{V^*}(g) \chi_W(g) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g).$$

According to Schur's Lemma, if V is irreducible, then $\dim(\mathrm{Hom}(V, W)^G)$ is the multiplicity of V in W. Since V, W are both irreducible, we have

$$\dim(\mathrm{Hom}(V, W)^G) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W \end{cases}.$$

□

The dimension of $\mathbb{C}_{class}(G)$ equals the number of conjugacy classes of G. This theorem tells us that the number of irreducible representations of G is bounded above by the number of conjugacy classes of G. In fact, the bound is optimal:

Theorem 2.4.10. *The characters of irreducible representations of G form an orthonormal basis of $\mathbb{C}_{class}(G)$.*

Proof. Let $\alpha \in \mathbb{C}_{class}(G)$. Suppose $\langle \alpha, \chi_{V_i} \rangle = 0$ for all irreducible representations V_i. We will show $\alpha = 0$. Consider the G-module homomorphism $\varphi_{\alpha,V} = \sum_{g \in G} \alpha(g) \cdot g : V \to V$. Consider the regular representation R of G. By Schur's lemma, $\varphi_{\alpha,R} = \lambda \cdot I$. Let $n = \dim R$, we have

$$\lambda = \frac{1}{n} \mathrm{Tr}(\varphi_{\alpha,R}) = \frac{1}{n} \sum \alpha(g) \cdot \chi_R(g) = \frac{|G|}{n} \overline{\langle \alpha, \chi_{R^*}(g) \rangle} = 0.$$

Hence, $\varphi_{\alpha,R} = \sum_{g \in G} \alpha(g) \cdot g = 0$. As elements in $\mathrm{End}(R)$, elements in G are linearly independent because $\{g(e) : g \in G\}$ are linearly independent. Hence, $\alpha(g) = 0$ for all g, i.e. $\alpha = 0$. □

Example 2.4.11. We will construct the character table of S_3. The group S_3 has three conjugacy classes, with representatives $1, (1\ 2)$ and $(1\ 2\ 3)$.

The 1-dimensional trivial representation and alternating representation are 1-dimensional, hence irreducible. We need one more irreducible representation.

Consider the permutation representation $\mathbb{C}^3$ of S_3, i.e. $\pi((x_1, x_2, x_3)) = (x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$. Decompose $\mathbb{C}^3$ into

$$\mathbb{C}^3 = \{(x, x, x) : x \in \mathbb{C}\} \oplus \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}.$$

We call the G-submodule $V_3 = \{(x_1, x_2, x_3) : x_1+x_2+x_3 = 0\}$ the *standard representation* of S_3. Notice that

$$\chi_{\mathbb{C}^3}(1) = 3, \chi_{\mathbb{C}^3}((1\ 2)) = 1, \chi_{\mathbb{C}^3}(1\ 2\ 3) = 0.$$

By lemma 2.4.3 $\mathbb{C}^3 \cong U_3 \oplus V_3$, we get

$$\chi_{V_3}(1) = 2, \chi_{V_3}((1\ 2)) = 0, \chi_{V_3}(1\ 2\ 3) = -1.$$

Since $\chi_{U_3}, \chi_{U_3'}$ and χ_{V_3} are orthonormal, we know V_3 is the third irreducible representation of S_3. We call V_3 the *standard representation of* S_3.

	1	3	2
S_3	1	$(1\ \ 2)$	$(1\ \ 2\ \ 3)$
U_3	1	1	1
U_3'	1	-1	1
V_3	2	0	-1

Table 2: Character Table of S_3

2.5 Restricted and Induced Representation

Definition 2.5.1 (restricted representation)**.** Let $H \leq G$ and X be a representation of G. Then the *restricted representation* of X to H, which is a representation of H, is given by $X{\downarrow_H}(h) = X(h)$.

Definition 2.5.2 (induced representation)**.** Let $H \leq G$ and Y be a representation of H. Let $g_1H, \ldots, g_nH$ be a traversal of left cosets of H. Then the *induced representation*, which is a representation of G, is given by $Y{\uparrow^G}(g) = (Y(g_i^{-1}gg_j))$, where define $Y(g)$ to be the zero matrix if $g \notin H$.

Example 2.5.3. Consider $G = S_3$ with subgroup $H = \{e, (1\ 2)\}$. Suppose Y is the trivial representation of H on $\mathbb{C}$, i.e., $Y(e) = Y((1\ 2)) = 1$. We will compute $Y{\uparrow^G}$.

Fix a traversal of left cosets of H: $\mathcal{H} = \{eH, (1\ 3)H, (2\ 3)H\}$. Then,

$$\begin{aligned}
Y{\uparrow^G}((2\ 3)) &= \begin{pmatrix} Y(e^{-1}(2\ 3)e) & Y(e^{-1}(2\ 3)(1\ 3)) & Y(e^{-1}(2\ 3)(2\ 3)) \\ Y((1\ 3)^{-1}(2\ 3)e) & Y((1\ 3)^{-1}(2\ 3)(1\ 3)) & Y((1\ 3)^{-1}(2\ 3)(2\ 3)) \\ Y((2\ 3)^{-1}(2\ 3)e) & Y((2\ 3)^{-1}(2\ 3)(1\ 3)) & Y((2\ 3)^{-1}(2\ 3)(2\ 3)) \end{pmatrix} \\
&= \begin{pmatrix} Y((2\ 3)) & Y((1\ 2\ 3)) & Y(e) \\ Y((1\ 3)) & Y((1\ 2)) & Y((1\ 3)) \\ Y(e) & Y((1\ 3)) & Y((2\ 3)) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Similarly, we can compute $Y{\uparrow^G}(g)$ for all $g \in S^3$.

Notice that Y coincides with the coset representation of G with respect to H in example 2.1.5.

Theorem 2.5.4. *Let $H \leq G$ and let 1 denote the trivial representation of H on $\mathbb{C}$. Then $1\uparrow^G$ is the same as the coset representation of G with respect to H.*

Proof. $(1\uparrow^G(g))_{i,j} = 1 \iff g_i^{-1} g g_j \in H \iff g g_j \in g_i H \iff g(g_j H) = g_i H.$ $\square$

3 Young Tableaux

In this section, we will introduce combinatorial objects, Young tableaux.

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ such that $\lambda_1 \geq \cdots \geq \lambda_l$ and $\lambda_1 + \cdots \lambda_l = n$, we can associate a *Young diagram*. For example, the Young diagram associate to the partition $(3, 1)$ is

$$\begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1}\end{array}$$

A Young tableau of shape $\lambda \vdash n$, denoted t^λ, is formed by filling in the Young diagram with $\{1, \ldots, n\}$. An example of $(3, 1)$-tableaux of is

$$\begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

A Young tableau with increasing rows and columns is called *standard*.

We now define an equivalence relation on the set of λ-tableaux: two λ-tableaux t_1, t_2 are *equivalent* if the corresponding rows of t_1, t_2 contain the same elements. Each equivalence class is called a λ-*tabloid*. For example, a complete list of $(2, 1)$-tabloids is

$$[t_1] = \begin{array}{cc} 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}, \quad [t_2] = \begin{array}{cc} 2 & 3 \\ \hline 1 & \\ \cline{1-1} \end{array}, \quad [t_3] = \begin{array}{cc} 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}.$$

Similarly, we can also obtain *column tabloids*, an equivalence class obtained by permuting the column of tableau.

If we relax our rules of filling in Young diagram by allowing repetitions, we get *generalized Young tableau*. An example of generalized Young tableaux is

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

The *content* of a generalized Young tableau T is an array recording the number of times an integer appears in T. For example, the content of the above generalized Young diagram is $(2, 2, 0, 1)$. A Young tableau with weakly increasing rows and strictly increasing columns are called *semistandard*.

4 Irreducible Representations of S_n

We've seen that the number of irreducible representations of G equals the number of conjugacy classes of G. In general, the correspondence is not well-understood. However, in the case of symmetric groups S_n, we can find an explicit correspondence via Young ttableaux.

Recall that conjugacy classes of S_n one-to-one correspond to partitions of n. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$,

$$S_\lambda := S_{\{1,\ldots,\lambda_1\}} \times \cdots \times S_{\{n-\lambda_l+1,\ldots,n\}} \cong S_{\lambda_1} \times \cdots \times S_{\lambda_l}$$

is called a *Young subgroup* of S_n. We can construct all irreducible representations of S_n using partitions $\lambda \vdash n$ as follows:

- *Step 1:* associate each S_λ an S_n-module M^λ;
- *Step 2:* find an ordering of partitions $\lambda^{(i)}$ such that
  - $S^{\lambda^{(1)}} := M^{\lambda^{(1)}}$ is irreducible;
  - for each i, $M^{\lambda^{(i)}} \cong M \oplus S^{\lambda^{(i)}}$ such that $S^{\lambda^{(i)}}$ is irreducible and the decomposition of M only contains $M^{\lambda^{(j)}}$ for some $j < i$.

In this section, we will examine this procedure in more details.

4.1 Construction of M^λ

We will give two equivalent ways of constructing M^λ in Step 1.

Let $S_\lambda \leq S_n$ and let 1 be the trivial representation of S_λ. Then inducing $1\uparrow_{S_\lambda}^{S_n}$ gives a representation. Recall from Section 2.5 that the induced representation $1\uparrow_{S_\lambda}^{S_n}$ is the left cosets representation of S_n. Specifically, fix a transversal $\mathbf{S}_\lambda = \{\pi_1 S_\lambda, \ldots, \pi_k S_\lambda\}$ of left cosets of S_λ, then $V^\lambda := \mathbb{C}[\mathbf{S}_\lambda]$ is an S_n-module.

Example 4.1.1. Consider $n = 3$ and $\lambda = (2, 1) \vdash 3$. Then, $S_\lambda = S_{\{1,2\}} \times S_{\{3\}} \cong \{e, (1\ 2)\}$ and $\mathbf{S}_\lambda = \{eH, (1\ 3)H, (2\ 3)H\}$. Notice that V^λ is the same as the induced representation in Example 2.5.3.

Let $\{t_i\}$ be a complete list of λ-tabloids. Since S_n acts on each element t_i by permuting the elements of t_i, then $M^\lambda := \mathbb{C}[\{t_i\}]$ is an S_n-module. We call M^λ a *permutation module*.

Example 4.1.2. Consider $n = 3$ and $\lambda = (2, 1)$. Then S_3 acts on $M^\lambda = \mathbb{C}[[t_1], [t_2], [t_3]]$. For example,

$$(2\ 3) \cdot \begin{array}{|cc|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} = \begin{array}{|cc|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}\ , \quad (2\ 3) \cdot \begin{array}{|cc|}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array} = \begin{array}{|cc|}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array}\ , \quad (2\ 3) \cdot \begin{array}{|cc|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} = \begin{array}{|cc|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

In matrix form,

$$(2\ 3) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Notice that this matrix coincides with the matrix in Example 2.5.3.

Theorem 4.1.3. *Let $\lambda \vdash n$. Then $V^\lambda \cong M^\lambda$ as S_n-modules.*

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_l)$. Suppose $\{t_i\}$ is a complete list of λ-tabloids, $\mathbf{S}_\lambda = \{\pi_1 S_\lambda, \ldots, \pi_k S_\lambda\}$, and

$$t_0 = \begin{array}{c} \overline{1 \quad 2 \quad 3 \quad \cdots \quad \lambda_1 - 1 \quad \lambda_1} \\ \vdots \\ \overline{n - \lambda_l + 1 \cdots \quad n} \end{array}.$$

Consider a $\mathbb{C}$-module homomorphism $\varphi : \mathbb{C}[\mathbf{S}_\lambda] \to \mathbb{C}[\{t_i\}]$ given by $\pi S_\lambda \mapsto \pi \cdot t_0$. Let $g \in S_n$ and $\pi S_\lambda \in V^\lambda$. Since the below diagram commutes, we know φ is a G-homomorphism.

$$\begin{tikzcd} V^\lambda \arrow[r, "\varphi"] \arrow[d, "g"'] & M^\lambda \arrow[d, "g"] \\ V^\lambda \arrow[r, "\varphi"] & M^\lambda \end{tikzcd}$$

Let t_i be a tabloid. There is a permutation π_i such that $\pi_i \cdot t_0 = t_i$. Then $\varphi(\pi_i S_\lambda) = t_i$, and thus φ is surjective.

Suppose $\varphi(\pi_j S_\lambda) = t_0$. Then π_j only permutes t_0 within each row, so $\pi_j \in S_\lambda$. Hence φ is injective. $\square$

4.2 Specht Module

The irreducible S_n-modules $S^{\lambda^{(i)}}$ in Step 2 are called Specht modules. We will construct Specht modules in this section.

Definition 4.2.1 (row-stabilizer; column-stabilizer)**.** Let t be a tableau with rows $R_1, \ldots, R_l$ and columns $C_1, \cdots, C_k$. Then the *row-stabilizer* is $R_t = S_{R_1} \times \cdots \times S_{R_l}$ and the *column-stabilizer* is $C_t = S_{C_1} \times \cdots \times S_{C_k}$.

Example 4.2.2. If $t = \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & \\ \cline{1-1} \end{array}$, then $R_t = S_{\{1,2\}} \times S_{\{3\}}$ and $C_t = S_{\{2,3\}} \times S_{\{1\}}$. Notice that the tabloid that t belongs to is $R_t \cdot t$.

Definition 4.2.3 (polytabloid)**.** Let t be a tableau. Then the associate *polytabloid* is $e_t = \kappa_t[t]$, where $[t]$ is the tabloid of t and $\kappa_t := \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi$.

Example 4.2.4. Continue with example 4.2.2, we have

$$\begin{aligned}
\kappa_t &= \sum_{\pi \in S_{\{2,3\}} \times S_{\{1\}}} \operatorname{sgn}(\pi)\pi \\
&= \sum_{\pi_i \in S_{\{2,3\}}} \sum_{\pi_j \in S_{\{1\}}} \operatorname{sgn}(\pi_i)\operatorname{sgn}(\pi_j)\pi_i\pi_j \\
&= \sum_{\pi_i \in S_{\{2,3\}}} \operatorname{sgn}(\pi_i)\pi_i \sum_{\pi_j \in S_{\{1\}}} \operatorname{sgn}(\pi_j)\pi_j \\
&= (e - (2\ 3))e = e - (2\ 3)
\end{aligned}$$

and hence

$$e_t = \begin{array}{|cc|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} - \begin{array}{|cc|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

Remark 4.2.5. In general, given tableau t, we have $\kappa_t = \kappa_{c_1} \cdots \kappa_{c_k}$ where $c_1, \ldots, c_k$ are columns of t.

Definition 4.2.6 (Specht module)**.** Let $\lambda \vdash n$. Then the corresponding Specht module S^λ is the submodule of M^λ spanned by the polytabloid e_t with t of shape λ.

Example 4.2.7. We will construct all irreducible representations of S_3 using Specht modules. We can partition 3 as $(3), (2,1)$ and (1^3), with corresponding Young diagrams

$$t^{(3)} = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array}\ , \quad t^{(2,1)} = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \\ \cline{1-1}\end{array}\ , \quad t^{(1^3)} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}\ .$$

Let's first construct $S^{(3)}$. The only Young tabloid of shape (3) is

$$t^{(3)} = \begin{array}{|ccc|}\hline 1 & 2 & 3 \\ \hline\end{array}.$$

We have

$$\kappa_t^{(3)} = \kappa_{c_1}^{(3)}\kappa_{c_2}^{(3)}\kappa_{c_3}^{(3)} = \sum_{\pi \in S_{\{1\}}} \operatorname{sgn}(\pi)\pi \sum_{\pi \in S_{\{2\}}} \operatorname{sgn}(\pi)\pi \sum_{\pi \in S_{\{3\}}} \operatorname{sgn}(\pi)\pi = e.$$

So we get

$$e_t^{(3)} = \begin{array}{|ccc|}\hline 1 & 2 & 3 \\ \hline\end{array}\ .$$

The corresponding Specht module $S^{(3)}$ is given by

$$S^{(3)} = \operatorname{span}(e_t^{(3)}) = \mathbb{C}[e_t^{(3)}].$$

Notice that $S^{(3)}$ is a 1-dimensional complex vector space that S_3 acts trivially on. Hence, $t^{(3)}$ corresponds to the irreducible trivial representation U_3 of S_3.

Let's now construct $S^{(2,1)}$. The three Young tabloid of $t^{(2,1)}$ is given in section 3. We have

$$\kappa_{t_1}^{(2,1)} = \sum_{\pi \in S_{\{1,3\}}} \operatorname{sgn}(\pi)\pi \sum_{\pi \in S_{\{2\}}} \operatorname{sgn}(\pi)\pi = (e - (1\ 3))e = e - (1\ 3)$$

$$\kappa_{t_2}^{(2,1)} = \sum_{\pi \in S_{\{1,2\}}} \operatorname{sgn}(\pi)\pi \sum_{\pi \in S_{\{3\}}} \operatorname{sgn}(\pi)\pi = (e - (1\ 2))e = e - (1\ 2)$$

$$\kappa_{t_3}^{(2,1)} = \sum_{\pi \in S_{\{1,2\}}} \operatorname{sgn}(\pi)\pi \sum_{\pi \in S_{\{3\}}} \operatorname{sgn}(\pi)\pi = (e - (1\ 2))e = e - (1\ 2)$$

It follows that

$$e_{t_1}^{(2,1)} = \begin{array}{|cc|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} - \begin{array}{|cc|}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array}$$

$$e_{t_2}^{(2,1)} = \begin{array}{|cc|}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array} - \begin{array}{|cc|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

$$e_{t_3}^{(2,1)} = \begin{array}{|cc|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} - \begin{array}{|cc|}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array}\ .$$

The corresponding Specht module $S^{(2,1)}$ is given by

$$S^{(2,1)} = \operatorname{span}(e_{t_1}^{(2,1)}, e_{t_2}^{(2,1)}, e_{t_3}^{(2,1)}) = \mathbb{C}[e_{t_1}^{(2,1)}, e_{t_2}^{(2,1)}]$$

Notice that $S^{(2,1)} = \{c_1[t_1] + c_2[t_2] + c_3[t_3] : c_1 + c_2 + c_3 = 0\}$; hence $S^{(2,1)}$ corresponds to the irreducible standard representation V_3 of S_3.

Consider the last Young diagram, $t^{(1^3)}$. There are six Young tabloids correspond to $t^{(1^3)}$. Consider

$$t_1^{(1^3)} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\ .$$

We have

$$\kappa_{t_1}^{(1^3)} = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\pi$$

and

$$e_{t_1}^{(1^3)} = \sum_{\pi \in S_n} \operatorname{sgn}(\pi)\pi[t_1^{(1^3)}]$$

Any of the other tabloids $[t_i^{(1^3)}]$ can be written as $\sigma \cdot [t_1^{(1^3)}]$ for some $\sigma \in S_n$. Since $e_{\sigma \cdot t_1}^{(1^3)} = \operatorname{sgn}(\sigma) e_{t_1}^{(1^3)}$, we have

$$S^{(1^3)} = \operatorname{span}(e_{t_1}^{(1^3)}) = \mathbb{C}[e_{t_1}^{(1^3)}].$$

Furthremore, the above observation shows $S^{(1^3)}$ corresponds to the alternating representation U_3' of S_3.

4.3 Complete list of irreducible S_n-modules

Our main goal in this section is to prove the S_n-modules S^λ are precisely the irreducible representations of S_n.

Lemma 4.3.1 (sign lemma)**.** *Let $H \leq S_n$ and denote $H^- = \sum_{\pi \in H} sgn(\pi)\pi$. Consider the inner product on M^λ given by $\langle u, v\rangle = \delta_{u,v}$.*

1. *for all $\pi \in H$, we have $\pi H^- = H^- \pi = sgn(\pi)H^-$;*
2. *for all $u, v \in M^\lambda$, we have $\langle H^- u, v\rangle = \langle u, H^- v\rangle$;*
3. *for all transpositions $(a\ b) \in H$, we have $H^- = k(e - (a\ b))$ for some $k \in \mathbb{C}[S_n]$;*
4. *for all tableau t such that a, b are in the same row and $(a\ b) \in H$, we have $H^-[t] = 0$.*

Proof. 1. $\pi H^- = \pi \sum_{\sigma \in H} \mathrm{sgn}(\sigma)\sigma = \sum_{\sigma \in H} \mathrm{sgn}(\sigma)\pi\sigma = \sum_{\sigma \in H} \mathrm{sgn}(\pi)\mathrm{sgn}(\pi\sigma)\pi\sigma = \mathrm{sgn}(\pi)H^-$.

2. $\langle H^- u, v\rangle = \sum_{\pi \in H} \mathrm{sgn}(\pi)\langle \pi u, v\rangle = \sum_{\pi \in H} \mathrm{sgn}(\pi)\langle u, \pi^{-1} v\rangle = \langle u, H^- v\rangle$.

3. Let k_i be representatives of $H/\{e, (a, b)\}$. Then $k = \sum_i \mathrm{sgn}(k_i)k_i$ suffices.

4. $H^-[t] = k(e - (a\ b))[t] = k([t] - [t]) = 0$.

□

We now introduce an ordering on partitions of n.

Definition 4.3.2 (dominance ordering of partitions)**.** Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_k)$ be partitions of n. We say λ *dominates* μ, denoted by $\lambda \trianglerighteq \mu$ if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all $i \geq 1$.

Lemma 4.3.3 (dominance lemma for partitions)**.** *Let $\lambda, \mu \vdash n$ and let t^λ, t^μ be tableaux of shape λ, μ such that the elements in each row of t^μ are in different columns in t^λ. Then $\lambda \trianglerighteq \mu$.*

Proof. Assume for each i, entires in first i rows of t^μ all occur in the first i rows of t^λ. Hence, $\mu_1 + \cdots \mu_i \leq$ (number of elements in the first i rows of t^λ) $= \lambda_1 + \cdots + \lambda_i$. □

Lemma 4.3.4. *Let $\lambda, \mu \vdash n$ and let t be a λ-tableau and let s be a μ-tableau. If $\kappa_t[s] \neq 0$, then $\lambda \trianglerighteq \mu$. If $\lambda = \mu$, then $\kappa_t[s] = \pm e_t$.*

Proof. Let a, b be two elements in the same row in s and in the same column of t. By the sign lemma, $\kappa_t = k(e - (a, b))$ for some $k \in \mathbb{C}[S_n]$ and $\kappa_t[s] = 0$. Hence, a, b are in different columns and $\lambda \trianglerighteq \mu$ by the dominance lemma.

Suppose $\mu = \lambda$. Since elements in the same row of s are in different column of t, there is some $\pi \in C_t$ such that $[s] = \pi[t]$. Hence, $\kappa_t[s] = \kappa_t \pi[t] = \mathrm{sgn}(\pi)\kappa_t[t] = \mathrm{sgn}(\pi)e_t$. □

Theorem 4.3.5. *Let $\lambda \vdash n$. Then $S^\lambda \subseteq M^\lambda$ is an irreducible S_n-module.*

Proof. Let U be a submodule of M^λ. Consider $u \in U$ and t a λ-tableau. Since $u = \sum_i c_i[t_i]$ for $c_i \in \mathbb{C}$ and λ-tableaux t_i, we have $\kappa_t u = \sum_i \pm c_i e_t$ by the previous lemma. In other words, for any t and $u \in U$, there exists $f \in \mathbb{C}[S_n]$ such that $\kappa_t u = f e_t$.

Suppose S^λ is not contained in U. Then for all $u \in U$ and t a λ-tableau, we have $\kappa_t u = 0$. We claim that $U \subseteq S^{\lambda\perp}$: let $u \in U$; for all λ-tableau t, $\langle u, e_t\rangle = \langle u, \kappa_t[t]\rangle = \langle \kappa_t u, [t]\rangle = \langle 0, [t]\rangle = 0$.

Therefore, S^λ is irreducible as $S^\lambda \cap S^{\lambda\perp} = \{0\}$. □

Theorem 4.3.6. *A complete list of irreducible S_n-module is given by S^λ for $\lambda \vdash n$.*

Proof. Suppose $S^\lambda \cong S^\mu$ for $\lambda, \mu \vdash n$. Then, there is a nonzero homomorphism $\varphi \in \mathrm{Hom}(S^\lambda, M^\mu)$. Let $e_t \in S^\lambda$ such that $\varphi(e_t) \neq 0$. Then, $\varphi(e_t) = \varphi(\kappa_t[t]) = \kappa_t\varphi([t]) = \kappa_t \sum_i c_i[s_i]$ for $c_i \in \mathbb{C}$ and μ-tableau s_i. By the previous lemma, $\lambda \trianglerighteq \mu$. Switching the role of λ and μ, we have $\mu \trianglerighteq \lambda$. Since the number of partitions of n equals the number of irreducible S_n module, S^λ forms a complete list of irreducible S_n-module. □

4.4 Basis of S^λ

Notice that the polytabloids generating S^λ might not be independent; for example, in example 4.2.7, we have $e_{t_1}^{(2,1)} + e_{t_2}^{(2,1)} = -e_{t_3}^{(2,1)}$. A natural question to ask is: what is a description of a basis of S^λ? It turns out the Specht modules indexed by standard tableaux form a basis of S^λ.

We introduce an ordering of tabloids. Let $[t]$ be a tabloid. For each $i \in \{1, \ldots, n\}$, let $[t^i]$ be the tabloid formed by elements smaller than or equal to i in $[t]$, and let λ^i denote the shape of $[t^i]$. We call λ^i a *composition sequence*.

Definition 4.4.1 (dominance ordering of tabloids)**.** Let $[s], [t]$ be two tabloids with composition sequences λ^i and μ^i. We say $[s]$ dominates $[t]$ if $\lambda^i \trianglerighteq \mu^i$ for all i.

Example 4.4.2. Consider $[s] = \begin{array}{cc} 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$ and $[t] = \begin{array}{cc} 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}$. Then

$$\lambda^1 = (1,0), \lambda^2 = (2,0), \lambda^3 = (2,1), \lambda^4 = (2,2)$$

$$\mu^1 = (0,1), \mu^2 = (0,2), \mu^3 = (1,2), \mu^4 = (2,2).$$

Hence $[s] \trianglerighteq [t]$.

Lemma 4.4.3 (dominance lemma for tabloids)**.** *Suppose $a < b$ and a is in a lower row than b in tableau t. Then $(a\ b)[t] \trianglerighteq [t]$.*

Proof. Suppose λ^i and μ^i are composition sequence of $(a\ b)[t]$ and $[t]$. Since $\lambda^i \trianglerighteq \mu^i$ for $a \leq i < b$ and $\lambda^i = \mu^i$ for $i < a$ or $i \geq b$, we have $(a\ b)[t] \trianglerighteq [t]$. □

Theorem 4.4.4. *The set $\{e_t : t$ is a standard λ-tableau$\}$ is linearly independent.*

Proof. Notice that a standard tabloid $[t]$ is the maximum element in e_t: let $s = \pi t$ for $\pi \in C_{t_i}$. We induct on the number of pairs $a < b$ in the same column of s such that a is in a lower row than b. By lemma 4.4.3, $(a\ b)[s] \trianglerighteq [s]$. By induction, $[t] \trianglerighteq (a\ b)[s] \trianglerighteq [s]$.

Suppose $\sum_i c_i e_{t_i} = 0$ for $c_i \in \mathbb{C}$. Assume $[t_1]$ is the maximal among the $[t_i]$. Since $[t_1]$ appears only in e_{t_i}, we have $c_1 = 0$. By induction, $c_i = 0$ for all i. □

Theorem 4.4.5. *The set $\{e_t : t$ is a standard λ-tableau$\}$ spans S^λ.*

Proof. Let $V = \text{span}\{e_t : t \text{ is a standard } \lambda\text{-tableau}\}$. We will do induction on tableaux. Suppose t_0 is the standard tabloid obtained by consecutively numbers the columns. Notice that $[t_0]$ is the maximum element of the poset of column tabloids. Since t_0 is standard, $[t_0] \in V$.

Consider a nonstandard tableau t with increasing column. Assume $[s] \in V$ for all $[s] \trianglerighteq [t]$. Suppose there is a descent in row i and $a_1 < a_2 < \cdots < a_p$, $b_1 < \cdots < b_q$ are the columns involved. Let $A = \{a_i, \ldots, a_p\}$, $B = \{b_1, \ldots, b_i\}$, and $g_{A,B} = \sum_\pi (\text{sgn}(\pi))\pi$ where π permutes t such that the elements in $A \cup B$ increase down the columns of πt. Then, $g_{A,B} e_t = 0$. Hence, $e_t = -\sum_{\pi \neq e} (\text{sgn}(\pi)) e_{\pi t}$. However, for all $\pi \neq e$, we have $\pi[t] \geq [t]$. Therefore, $e_t \in V$. □

4.5 Decomposition of M^μ

The goal of this section is to find the decomposition of M^μ in terms of Specht modules. First, we will give another construction of M^μ. Let $\mathcal{T}_{\lambda\mu}$ be the set of generalized Young tableaux with shape λ and content μ. Then for any partitions λ, $M^\mu \cong \mathbb{C}[\mathcal{T}_{\lambda\mu}]$.

Theorem 4.5.1 (Young's Rule)**.** *Let $\mu \vdash n$ and M^μ be a permutation module. Then $M^\mu \cong \oplus_{\lambda \trianglerighteq \mu} K_{\lambda\mu} S^\lambda$, where $K_{\lambda\mu}$ is the number of semistandard tableaux of shape λ and content μ.*

Proof. Let T be a generalized tableau. Let $\{T\}$ and $[T]$ denote the row and column equivalence classes of T. Suppose $\mathcal{B} = \{\theta_T|_{S^\lambda} : T \in \mathcal{T}^0_{\lambda\mu}\}$ where $\mathcal{T}^0_{\lambda\mu}$ is the set of semistandard λ-tableaux of content μ and $\theta : M^\lambda \to M^\mu$ is given by $[t] \mapsto \sum_{S \in \{T\}} S$. We claim that $\mathcal{B}$ is a basis for $\text{Hom}(S^\lambda, M^\mu)$. The proof of the claim is parallel to the proof of basis of S^λ in the previous section. Since the number of copies of S^λ in M^μ is the dimension of $\text{Hom}(S^\lambda, M^\mu)$, $M^\lambda \cong \oplus_{\lambda \trianglerighteq \mu} K_{\lambda\mu} S^\lambda$. □

Example 4.5.2. Consider $\mu = (1,1,1)$ and S_3-module M^μ. Partitions λ with $\lambda \vdash \mu$ are $(1,1,1), (2,1)$ and (3). The λ-tableaux of content $(1,1,1)$ are

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

So we get

$$M^{(1,1,1)} \cong S^{(1,1,1)} \oplus 2S^{(2,1)} \oplus S^{(3)}.$$

5 Grassmannian

5.1 Schubert Variety

Young diagrams are closely related to intersection of Schubert varieties. In this section, we gave a very breif introduction of the connection between Young diagrams and Schubert Calculus.

Definition 5.1.1 (Grassmannian)**.** Suppose V is a n-dimensional complex vector space. The *Grassmannian* $Gr(k,n)$ is the set of all k-dimensional subspaces of V.

Theorem 5.1.2. *$Gr(k,n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$ is an irreducible projective variety in $\mathbb{P}^{\binom{n}{k}-1}$ given by Plücker relations.*

Example 5.1.3. $\mathrm{Gr}(2,4)$ is the set of complex planes in $\mathbb{C}^4$. If we denote the coordinates of $\mathbb{P}^5$ by $[x_{12} : x_{13} : x_{14} : x_{23} : x_{24} : x_{34}]$, then $\mathrm{Gr}(2,4)$ embeds in $\mathbb{P}^5$ as a variety, cutout by $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$.

Definition 5.1.4 (Schubert cell)**.** Suppose V is a n-dim complex vectorspace with basis $v_1, \ldots, v_n$. Let $V_m = \mathrm{span}\{v_1, \ldots, v_m\}$ for $m \in \{1, \ldots, n\}$, and let λ be a partition inside rectangle $k \times (n-k)$. A Schubert cell indexed by $\lambda = (\lambda_1, \ldots, \lambda_k)$ is $\Omega_\lambda = \{U \in Gr(k,V) : \dim(U \cap V_j) = i \iff n-k+i-\lambda_i \leq j \leq n-k+i-\lambda_{i+1}\}$.

Example 5.1.5. Consider $Gr(2,4)$, the set of planes in $\mathbb{C}^4$. $Gr(2,4)$ decomposes into six Schubert cells, indexed by the following six Young diagrams:

$$\emptyset\ ,\ \ydiagram{1}\ ,\ \ydiagram{2}\ ,\ \ydiagram{1,1}\ ,\ \ydiagram{2,1}\ ,\ \ydiagram{2,2}\ .$$

Let $U = \mathrm{span}\{2v_1 + v_2, v_3 + v_4\} \in \mathrm{Gr}(2,4)$. We have $\dim(U \cap V_1) = 0, \dim(U \cap V_2) = 1, \dim(U \cap V_3) = 1, \dim(U \cap V_4) = 2$. Hence, $U \in \Omega_{(1,0)}$. Moreover, elements in $\Omega_{(1,0)}$ are the row space of the matrix of form $\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{pmatrix}$.

Definition 5.1.6 (Schubert variety)**.** A Schubert variety indexed by $\lambda = (\lambda_1, \ldots, \lambda_k)$ is $X_\lambda = \{U \in Gr(k,n) : \dim(U \cap V_{n-k+i-\lambda_i}) \geq i\}$.

Theorem 5.1.7. *Let λ, μ be a partition inside rectangle $k \times (n-k)$. Then $X_\lambda = \overline{\Omega_\lambda}$ and $X_\lambda \supseteq X_\mu \iff \lambda \subseteq \mu$.*

Definition 5.1.8 (Schubert class)**.** Let X be a Schubert variety. Then the basis element of the cohomology ring induced by X is called Schubert class, denoted by $[X]$.

Theorem 5.1.9. *Suppose X_i, X_j are Schubert varieties in generic positions. Then $[X_i] \cup [X_j] = [X_i \cap X_j]$.*

5.2 An Example

Example 5.2.1. A classical problem in enumerative geometry is "to find the number of lines meeting four given lines in a three-dimensional space". We avoid dealing with the dependence of answer on the specific configuration of lines, and we answer the question "find the number of lines meeting four gives lines in generic positions in $\mathbb{P}^3$".

An equivalent question in terms of Grassmannians is "to find the number of elements in $Gr(2,4)$ that are in the intersection of four copies of $X_{(1,0)}$".

The answer is given by

$$[X_{(1,0)} \cap X_{(1,0)} \cap X_{(1,0)} \cap X_{(1,0)}] = [X_{(1,0)}] \cup [X_{(1,0)}] \cup [X_{(1,0)}] \cup [X_{(1,0)}] = 2[X_{(2,2)}]$$

where the second equality is given by multiplication of Young diagrams with Littlewood-Richardson rule:

$$X_{\ydiagram{1}} \cdot X_{\ydiagram{1}} \cdot X_{\ydiagram{1}} \cdot X_{\ydiagram{1}} = (X_{\ydiagram{2}} + X_{\ydiagram{1,1}}) \cdot X_{\ydiagram{1}} \cdot X_{\ydiagram{1}} = (X_{\ydiagram{2,1}} + X_{\ydiagram{2,1}}) \cdot X_{\ydiagram{1}} = 2 \cdot X_{\ydiagram{2,2}}$$

Therefore, four given lines in generic position intersect at two points.

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